Logic

(Propositional Logic)

REPRESENTING KNOWLEDGE: LOGIC

- Logic is the branch of mathematics / philosophy concerned with knowledge and reasoning
- Aristotle distinguished between three types of arguments:
  - LOGIC arguments: that produce true conclusions from true premisses
  - DIALECTIC arguments: that produce true conclusions from plausible premisses
  - SOPHISTIC arguments: incorrect

VALID ARGUMENTS: CONNECTIVES

If students like AI, Folajimi is happy

Students like AI

Folajimi is happy

VALID ARGUMENTS: QUANTIFIERS

Birds fly

Swallows are birds

Swallows fly

INVALID ARGUMENTS (FALLACIES)

If students like AI, Folajimi is happy

Folajimi is happy

Students like AI

TWO MAIN FORMS OF LOGIC

- PROPOSITIONAL CALCULUS
  - Valid arguments involving CONNECTIVES
  - Propositions remain unanalyzed
- PREDICATE CALCULUS
  - Analyze propositions into PREDICATES and ARGUMENTS
  - This makes it possible to study valid arguments involving QUANTIFIERS, as well (FIRST ORDER LOGIC)
  - (A generalization of Syllogism logic)

CHARACTERISTICS OF A MODERN LOGIC

- VOCABULARY: the set of SYMBOLS in the language
- SYNTAX: a set of rules to combine symbols into phrases
- SEMANTICS: the interpretation of the symbols and the phrases
- A PROOF THEORY: a system of formal rules to derive formulas from other formulas
  - In classical logic, preserving VALIDITY

PROPOSITIONAL CALCULUS

- The logic of CONNECTIVES: and, or, not, if … then
- Originally formulated by the Stoics (Crisippo)

REPRESENTING KNOWLEDGE IN LOGIC: PROPOSITIONAL CALCULUS

- p: students like AI
- q: Folajimi is happy
- $p \rightarrow q$?

- r: Every bicycle has 2 wheels

THE CONNECTIVES: CONJUNCTION

- p: CSC is in UI
- q: CSC has 500 students
- p & q: CSC is in UI
and
(CSC) has 500 students
- CSC and Chemistry are in UI
- Catering Department has offices in Awo and Mellamby

THE CONNECTIVES: NEGATION

- p: The University of Ibadan has a Faculty of Science
- ~p: It is not the case that the University of Ibadan has a Faculty of Science
- ~p: The University of Ibadan does not have a Faculty of Science
- ~p: There is no Faculty of Science at the University of Ibadan

THE CONNECTIVES: DISJUNCTION

- p: CSC has 500 students
- q: CSC has 700 students
- $p \vee q$: CSC has 500
  or
  (CSC has) 700 students
- Folajimi studied in Nigeria or USA

THE CONNECTIVES: IMPLICATION

- If students like AI, Folajimi is happy
  - Two clear cases:
    - If the students like AI and Folajimi is happy, the implication is true
    - If the students like AI, but Folajimi is not happy, the implication is false
  - There are also difficult cases:
    - If the students don’t like AI, is the implication true or false?
    - Convention : yes!
    - Argument: implication does not make any claim at all about these cases)
    - Unpleasant consequence: If 2+2=5, I am the Lecturer: true!!

THE CONNECTIVES: BICONDITIONAL

- I will go walking if I get my car fixed
  - IMPLICATION: only false if I get my car fixed but then I don't go walking. OK if I go walking even if I don't get my car fixed
  - Often more intuitive if reverse: If I get my car fixed, I will go walking.
- I will go walking if, and only if, I get my car fixed
  - BICONDITIONAL: if I don't get my car fixed I don't go walking
  - Getting the car fixed is a NECESSARY and SUFFICIENT condition to go walking

Propositional Logic

Propositional logic is concerned with the truth or falsehood of statements (propositions) like:

the valve is closed

five plus four equals nine

Connectives:		
	and	$\wedge$
	or	$\vee$
	not	$\neg$
	implies	$\rightarrow$
	equivalent	$\leftrightarrow$

$$(X \rightarrow (Y \wedge Z)) \leftrightarrow ((X \rightarrow Y) \wedge (X \rightarrow Z))$$

"X implies Y and Z is the same as X implies Y and X implies Z"

Propositional Logic

- Propositional Logic is declarative, pieces of syntax correspond to facts
- Propositional Logic is compositional
  - Meaning of $A \wedge B$ derived from meaning of A and B
- Meaning in Propositional Logic is context-independent
- Propositional Logic has very limited expressive power
  - Cannot say "*Scottish men are careful with money*"

Propositional logic

- **Logical constants**: true, false
- **Propositional symbols**: P, Q, S, ...
- Wrapping **parentheses**: (…)
- Sentences are combined by **connectives**:

  $\wedge$ ...and

  $\vee$ ...or

  $\Rightarrow$...implies

  $\Leftrightarrow$..is equivalent

  $\neg$ ...not

Propositional logic (PL)

- A simple language useful for showing key ideas and definitions
- User defines a set of propositional symbols, like P and Q.
- User defines the semantics of each of these symbols, e.g.:
  - P means "It is hot"
  - Q means "It is humid"
  - R means "It is raining"
- A sentence (aka formula, well-formed formula, wff) defined as:
  - A symbol
  - If S is a sentence, then ~S is a sentence (e.g., "not")
  - If S is a sentence, then so is (S)
  - If S and T are sentences, then (S v T), (S ^ T), (S => T), and (S <=> T) are sentences (e.g., "or," "and," "implies," and "if and only if")
  - A finite number of applications of the above

Examples of PL sentences

- (P ^ Q) => R

  "If it is hot and humid, then it is raining"
- Q => P

  "If it is humid, then it is hot"
- Q

  "It is humid."
- A better way:

  Ho = "It is hot"

  Hu = "It is humid"

  R = "It is raining"

A BNF grammar of sentences in propositional logic

```
S := <Sentence> ;
<Sentence> := <AtomicSentence> |
 <ComplexSentence> ;
<AtomicSentence> := "TRUE" | "FALSE" |
                    "P" | "Q" | "S" ;
<ComplexSentence> := "(" <Sentence> ")" |
           <Sentence> <Connective> <Sentence>
 |
           "NOT" <Sentence> ;
<Connective> := "NOT" | "AND" | "OR" |
 "IMPLIES" | "EQUIVALENT" ;
```

Some terms

- The meaning or **semantics** of a sentence determines its **interpretation**.
- Given the truth values of all of symbols in a sentence, it can be "evaluated" to determine its **truth value** (True or False).
- A **model** for a KB is a "possible world" in which each sentence in the KB is True.
- A **valid sentence** or **tautology** is a sentence that is True under all interpretations, no matter what the world is actually like or what the semantics is. Example: "It's raining or it's not raining."
- An **inconsistent sentence** or **contradiction** is a sentence that is False under all interpretations. The world is never like what it describes, as in "It's raining and it's not raining."
- **P entails Q**, written $P \models Q$, means that whenever P is True, so is Q. In other words, all models of P are also models of Q.

Truth tables

And

p	q	$p \cdot q$
T	T	T
T	F	F
F	T	F
F	F	F

Or

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

If . . . then

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Not

p	$\sim p$
T	F
F	T

A bit more about =>

- Isn't it strange that P=>Q is true whenever P is false?
  - Consider P:"if you try" and Q:"you will succeed". P=>Q : "if you try then you will succeed"
  - Obviously if P and Q are true, P=>Q is true and if P is true and Q is false then P=>Q is false
  - But if P is false (i.e. you don't try) then there is no way we can tell that P=>Q is false. So it must be true
    - There is no such thing as "Unknown" value in propositional logic

Truth tables II

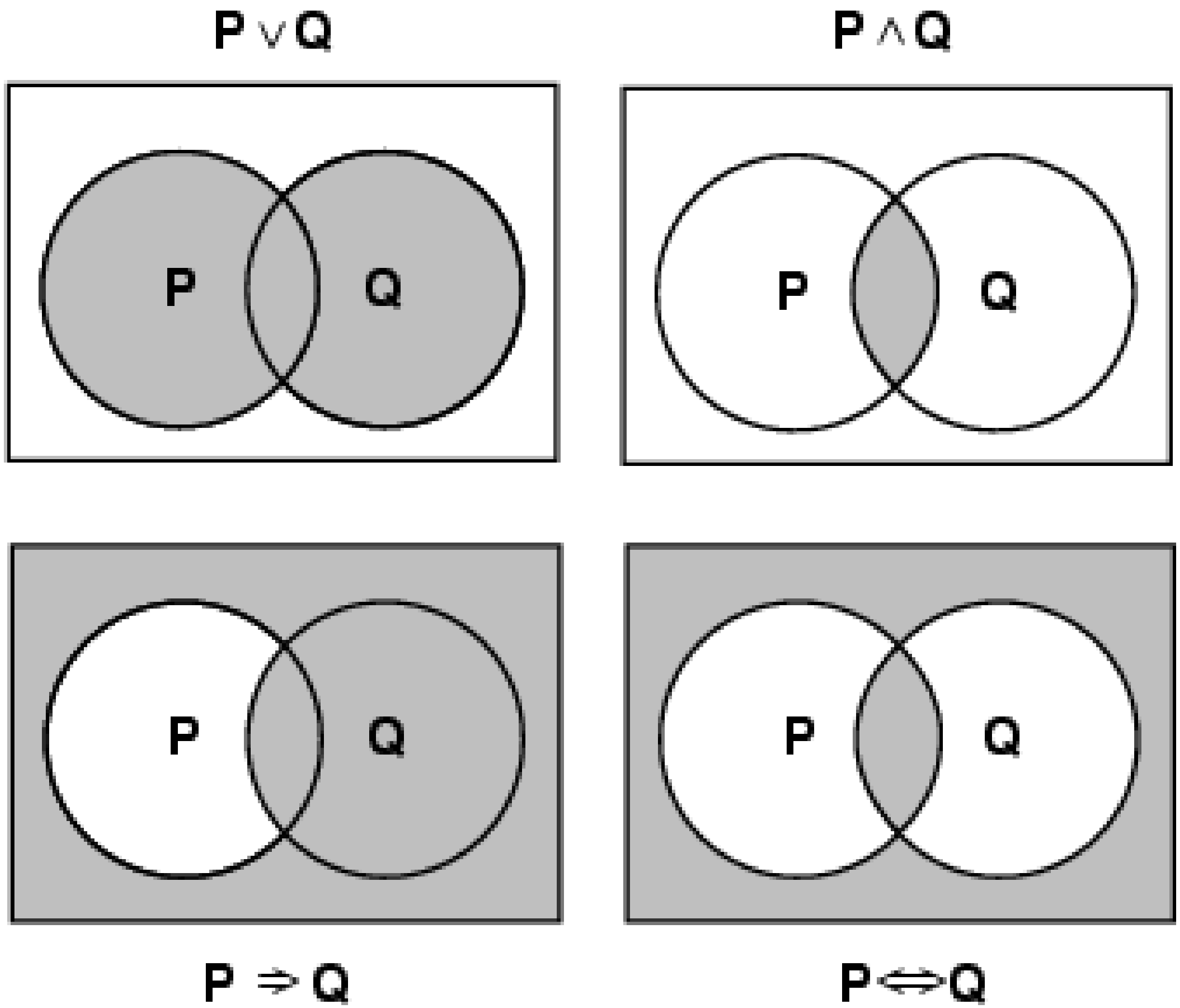
The five logical connectives:

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
False	*False*	*True*	*False*	*False*	*True*	*True*
False	*True*	*True*	*False*	*True*	*True*	*False*
True	*False*	*False*	*False*	*True*	*False*	*False*
True	*True*	*False*	*True*	*True*	*True*	*True*

A complex sentence:

P	H	$P \vee H$	$(P \vee H) \wedge \neg H$	$((P \vee H) \wedge \neg H) \Rightarrow P$
False	*False*	*False*	*False*	*True*
False	*True*	*True*	*False*	*True*
True	*False*	*True*	*True*	*True*
True	*True*	*True*	*False*	*True*

Models of complex sentences



Inference rules

- **Logical inference** is used to create new sentences that logically follow from a given set of predicate calculus sentences (KB).
- An inference rule is **sound** if every sentence X produced by an inference rule operating on a KB logically follows from the KB. (That is, the inference rule does not create any contradictions)
- An inference rule is **complete** if it is able to produce every expression that logically follows from (is entailed by) the KB. (Note the analogy to complete search algorithms.)

Sound rules of inference

- Here are some examples of sound rules of inference.
- Each can be shown to be sound using a truth table: A rule is sound if its conclusion is true whenever the premise is true.

RULE	PREMISE	CONCLUSION
Modus Ponens	A, A => B	B
And Introduction	A, B	A ^ B
And Elimination	A ^ B	A
Double Negation	~~A	A
Unit Resolution	A v B, ~B	A
Resolution	A v B, ~B v C	A v C

Soundness of modus ponens

A	B	$A \rightarrow B$	OK?
True	True	True	√
True	False	False	√
False	True	True	√
False	False	True	√

Soundness of the resolution inference rule

α	β	γ	$\alpha \vee \beta$	$\neg\beta \vee \gamma$	$\alpha \vee \gamma$
False	*False*	*False*	*False*	*True*	*False*
False	*False*	*True*	*False*	*True*	*True*
False	*True*	*False*	*True*	*False*	*False*
False	*True*	*True*	*True*	*True*	*True*
True	*False*	*False*	*True*	*True*	*True*
True	*False*	*True*	*True*	*True*	*True*
True	*True*	*False*	*True*	*False*	*True*
True	*True*	*True*	*True*	*True*	*True*

Proving Things

- **Consider a KB consisting of the following rules and facts (collectively called premises)**
- (Ho ^ Hu) => R

  "If it is hot and humid, then it is raining"
- Hu => Ho

  "If it is humid, then it is hot"
- Hu

  "It is humid."

  We want to prove that it is raining

Proving things

- A **proof** is a sequence of sentences, where each sentence is either a premise or a sentence derived from earlier sentences in the proof by one of the rules of inference.
- The last sentence is the **theorem** (also called goal or query) that we want to prove.
- Example for the "weather problem" given above.

1 Hu	Premise	"It is humid"
2 Hu=>Ho	Premise	"If it is humid, it is hot"
3 Ho	Modus Ponens(1,2)	"It is hot"
4 (Ho^Hu)=>R	Premise	"If it's hot & humid, it's raining"
5 Ho^Hu	And Introduction(1,2)	"It is hot and humid"
6 R	Modus Ponens(4,5)	"It is raining"

Equivalences in PL

idempotency laws

$$A \wedge A \simeq A$$
$$A \vee A \simeq A$$

commutative laws

$$A \wedge B \simeq B \wedge A$$
$$A \vee B \simeq B \vee A$$

associative laws

$$(A \wedge B) \wedge C \simeq A \wedge (B \wedge C)$$
$$(A \vee B) \vee C \simeq A \vee (B \vee C)$$

distributive laws

$$A \vee (B \wedge C) \simeq (A \vee B) \wedge (A \vee C)$$
$$A \wedge (B \vee C) \simeq (A \wedge B) \vee (A \wedge C)$$

de Morgan laws

$$\neg(A \wedge B) \simeq \neg A \vee \neg B$$
$$\neg(A \vee B) \simeq \neg A \wedge \neg B$$

other negation laws

$$\neg(A \to B) \simeq A \wedge \neg B$$
$$\neg(A \leftrightarrow B) \simeq (\neg A) \leftrightarrow B \simeq A \leftrightarrow (\neg B)$$

laws for eliminating certain connectives

$$A \leftrightarrow B \simeq (A \to B) \wedge (B \to A)$$
$$\neg A \simeq A \to \mathbf{f}$$
$$A \to B \simeq \neg A \vee B$$

simplification laws

$$A \wedge \mathbf{f} \simeq \mathbf{f}$$
$$A \wedge \mathbf{t} \simeq A$$
$$A \vee \mathbf{f} \simeq A$$
$$A \vee \mathbf{t} \simeq \mathbf{t}$$
$$\neg\neg A \simeq A$$
$$A \vee \neg A \simeq \mathbf{t}$$
$$A \wedge \neg A \simeq \mathbf{f}$$

Note: Proof using Truth Table

Duality Principle

- Propositional logic enjoys a principle of duality:
  - for every equivalence A ⇔ B there is another equivalence A' ⇔ B' where A', B' are derived from A, B by exchanging ^ with v and t with f. Before applying this rule, remove all occurrences of →and ⇔, since they implicitly involve ^ and v.

Entailment and derivation

- **Entailment: KB $\models$ Q**
  - Q is entailed by KB (a set of premises or assumptions) if and only if there is no logically possible world in which Q is false while all the premises in KB are true.
  - Or, stated positively, Q is entailed by KB if and only if the conclusion is true in every logically possible world in which all the premises in KB are true.
- **Derivation: KB $\vdash$ Q**
  - We can derive Q from KB if there is a proof consisting of a sequence of valid inference steps starting from the premises in KB and resulting in Q

Two important properties for inference

Soundness: If KB $\vdash$ Q then KB $\models$ Q

- If Q is derived from a set of sentences KB using a given set of rules of inference, then Q is entailed by KB.
- Hence, inference produces only real entailments, or any sentence that follows deductively from the premises is valid.

Completeness: If KB $\models$ Q then KB $\vdash$ Q

- If Q is entailed by a set of sentences KB, then Q can be derived from KB using the rules of inference.
- Hence, inference produces all entailments, or all valid sentences can be proved from the premises.

Definitions: Normal Forms

- A literal is an atomic formula or its negation. Let K, L, L', … stand for literals.
- A formula is in Negation Normal Form (NNF) if the only connectives in it are ^, v, and ┐, where ┐ is only applied to atomic formulae.
- A formula is in Conjunctive Normal Form (CNF) if it has the form A_1 ^ … ^ A_m, where each A_i is a disjunction of one or more literals.
- A formula is in Disjunctive Normal Form (DNF) if it has the form A_1 v … vA_m, where each A_i is a conjunction of one or more literals.

Normal Forms

- An atomic formula like P is in all the normal forms NNF, CNF, and DNF. The formula $(P \vee Q) \wedge (\neg P \vee S) \wedge (R \vee P)$ is in CNF
- Simplifying the formula $(P \vee Q) \wedge (\neg P \vee Q) \wedge (R \vee S)$ to $Q \wedge (R \vee S)$ counts as an improvement.
- Converting $\neg(A \rightarrow B)$ to NNF yields $A \wedge \neg B$.
- Every formula in CNF or DNF is also in NNF, but the NNF formula $((\neg P \wedge Q) \vee R) \wedge P$ is neither CNF or DNF

Translation to normal form

- Every formula can be translated into an equivalent formula in NNF, CNF, or DNF
- Step 1. Eliminate →and ⇔ by repeatedly applying the following equivalences:

$$A \leftrightarrow B \simeq (A \rightarrow B) \wedge (B \rightarrow A)$$

$$A \rightarrow B \simeq \neg A \vee B$$

- Step 2. Push negations in until they apply only to atoms, repeatedly replacing by the equivalences

$$\neg\neg A \simeq A$$

$$\neg(A \wedge B) \simeq \neg A \vee \neg B$$

$$\neg(A \vee B) \simeq \neg A \wedge \neg B$$

- At this point, the formula is in Negation Normal Form.

Translation to normal form (contd.)

- Step 3. To obtain CNF, push disjunctions in until they apply only to literals. Repeatedly replace by the equivalences

$$A \vee (B \wedge C) \simeq (A \vee B) \wedge (A \vee C)$$

$$(B \wedge C) \vee A \simeq (B \vee A) \wedge (C \vee A)$$

  –

- Step 4. Simplify the resulting CNF by deleting any disjunction that contains both P and $\neg P$, since it is equivalent to t. Also delete any conjunct that includes another conjunct (meaning, every literal in the latter is also present in the former). This is correct because $(A \vee B) \wedge A \simeq A.$
- Finally, two disjunctions of the form $P \vee A$ and $\neg P \vee A$ can be replaced by A, thanks to the equivalence

$$(P \vee A) \wedge (\neg P \vee A) \simeq A.$$

Translation to normal form (contd.)

- Steps 3' and 4'. To obtain DNF, apply instead the other distributive law:

$$A \wedge (B \vee C) \simeq (A \wedge B) \vee (A \wedge C)$$

$$(B \vee C) \wedge A \simeq (B \wedge A) \vee (C \wedge A)$$

- Exactly the same simplifications can be performed for DNF as for CNF, exchanging
- the roles of ^ and v.

Tautology checking using CNF

- Here is a method of proving theorems in propositional logic. To prove A, reduce it to CNF. If the simplified CNF formula is t then A is valid because each transformation preserves logical equivalence. And if the CNF formula is not t, then A is not valid.
- **Proof:**
  - suppose the CNF formula is $A_1 \wedge \ldots \wedge A_m$. If A is valid then each A_i must also be valid. Write A_i as $L_1 \vee \ldots \vee L_n$, where the L_j are literals. We can make an interpretation I that falsifies every L_j, and therefore falsifies A_i.
  - Define I such that, for every propositional letter P,

$$I(P) = \begin{cases} \mathbf{f} & \text{if } L_j \text{ is } P \text{ for some } j \\ \mathbf{t} & \text{if } L_j \text{ is } \neg P \text{ for some } j \end{cases}$$

  - This definition is legitimate because there cannot exist literals L_j and L_k such that L_j is $\neg L_k$; if there did, then simplification would have deleted the disjunction Ai .

Examples

Example 1 Start with

$$P \vee Q \rightarrow Q \vee R$$

Step 1, eliminate $\rightarrow$, gives

$$\neg(P \vee Q) \vee (Q \vee R)$$

Step 2, push negations in, gives

$$(\neg P \wedge \neg Q) \vee (Q \vee R)$$

Step 3, push disjunctions in, gives

$$(\neg P \vee Q \vee R) \wedge (\neg Q \vee Q \vee R)$$

Simplifying yields

$$(\neg P \vee Q \vee R) \wedge \mathbf{t}$$

$$\neg P \vee Q \vee R$$

The interpretation $P \mapsto \mathbf{t},\ Q \mapsto \mathbf{f},\ R \mapsto \mathbf{f}$ falsifies this formula, which is equivalent to the original formula. So the original formula is not valid.

Examples

Example 2 Start with

$$P \wedge Q \rightarrow Q \wedge P$$

Step 1, eliminate $\rightarrow$, gives $\neg(P \wedge Q) \vee Q \wedge P$

Step 2, push negations in, gives

$$(\neg P \vee \neg Q) \vee (Q \wedge P)$$

Step 3, push disjunctions in, gives

$$(\neg P \vee \neg Q \vee Q) \wedge (\neg P \vee \neg Q \vee P)$$

Simplifying yields t ^ t, which is t. Both conjuncts are valid since they contain a formula and its negation. Thus $P \wedge Q \rightarrow Q \wedge P$ is valid.

Propositional logic is a weak language

- Hard to identify "individuals." E.g., Mary, 3
- Can't directly talk about properties of individuals or relations between individuals. E.g. "Bill is tall"
- Generalizations, patterns, regularities can't easily be represented. E.g., all triangles have 3 sides
- First-Order Logic (abbreviated FOL or FOPC) is expressive enough to concisely represent this kind of situation.

FOL adds relations, variables, and quantifiers, e.g.,

- *"Every elephant is gray":* $\forall$ x (elephant(x) → gray(x))
- *"There is a white alligator":* $\exists$ x (alligator(X) ^ white(X))

Example

- Consider the problem of representing the following information:
  - Every person is mortal.
  - Confucius is a person.
  - Confucius is mortal.
- How can these sentences be represented so that we can infer the third sentence from the first two?

Example II

- In PL we have to create propositional symbols to stand for all or part of each sentence. For example, we might do:

  P = “person”; Q = “mortal”; R = “Confucius”

- so the above 3 sentences are represented as:

  P => Q; R => P; R => Q

- Although the third sentence is entailed by the first two, we needed an explicit symbol, R, to represent an individual, Confucius, who is a member of the classes “person” and “mortal.”
- To represent other individuals we must introduce separate symbols for each one, with means for representing the fact that all individuals who are “people” are also "mortal.”

Summary (so far)

- The process of deriving new sentences from old ones is called **inference**.
  - **Sound** inference processes derives true conclusions given true premises.
  - **Complete** inference processes derive all true conclusions from a set of premises.
- A **valid sentence** is true in all worlds under all interpretations.
- If an implication sentence can be shown to be valid, then - given its premise - its consequent can be derived.
- Different logics make different **commitments** about what the world is made of and what kind of beliefs we can have regarding the facts.
  - Logics are useful for the commitments they do not make because lack of commitment gives the knowledge base writer more freedom.

Summary (so far)

- **Propositional logic** commits only to the existence of facts that may or may not be the case in the world being represented.
  - It has a simple syntax and a simple semantic. It suffices to illustrate the process of inference.
  - Propositional logic quickly becomes impractical, even for very small worlds.

First-order Logic

- Whereas Propositional Logic assumes the world contains facts, First-order Logic assumes the world contains:
  - Objects; people, houses, games, wars, colours…
  - Relations; red, round, bogus, prime, brother, bigger_than, part_of…
  - Functions; father_of, best_friend, second_half_of, number_of_wheels…

Aside: Logics in General

Language	Ontological Commitment	Epistemological Commitment
Proposotional Logic	facts	true/false/unknown
First-order Logic	facts, objects, relations	true/false/unknown
Temporal Logic	facts, objects, relations, times	true/false/unknown
Probabilistic Logic	facts	degree of belief $\in$[0, 1]
Fuzzy Logic	degrees of truth $\in$[0, 1]	known interval value

Aside: Ontology & Epistemology

- **ontology** - *Ontology is the study of what there is, an inventory of what exists. An ontological commitment is a commitment to an existence claim.*
  - From Dictionary of Philosophy of Mind - **ontology**
    - http://www.artsci.wustl.edu/~philos/MindDict/ontology.html
  - Ontological Commitment:
    - What exists in the world
- **epistemology** - *A major branch of philosophy that concerns the forms, nature, and preconditions of knowledge.*
  - Epistemological Commitment:
    - What an agent believes about facts

Predicate Calculus

Two important extensions to Propositional Logic are:

predicates and quantifiers

- Predicates are statements about objects by themselves or in relation to other objects. Some examples are:

  less-than-zero

  weighs-more-than

- The quantifiers then operate over the predicates. There are two quantifiers:

  $\forall$ for all (the universal quantifier)

  $\exists$ there exists (the existential quantifier)

So with predicate calculus we can make statements like:

$\forall$ XYZ: Smaller(X,Y) $\wedge$ Smaller(X,Z) $\rightarrow$ Smaller(X,Z)

Predicate Calculus (cont'd)

Predicate calculus is very general but not very powerful

Two useful additions are functions and the predicate equals

absolute-value

number-of-wheels

colour

First Order Logic

Two individuals are "equal" if and only if they are indistinguishable under all predicates and functions.

Predicate calculus with these additions is a variety of *first order logic.*

A logic is of *first order* if it permits quantification over individuals but not over predicates and functions. For example a statement like "all predicates have only one argument" cannot be expressed in first order logic.

The advantage of first order logic as a representational formalism lies in its formal structure. It is relatively easy to check for *consistency* and *redundancy.*

First Order Logic

- Advantages stem from rigid mathematical basis for first order logic
- This rigidity gives rise to problems. The main problem with first order logic is that it is monotonic.

"If a sentence S is a logical consequence of a set of sentences A then S is still a logical consequence of any set of sentences that includes A. So if we think of A as embodying the set of beliefs we started with, the addition of new beliefs cannot lead to the *logical* repudiation of old consequences."

So the set of theorems derivable from the premises is not reduced (increases monotonically) by the adding of new premises.