First Order Logic

First order logic

- In the propositional logic, the most basic elements are atoms.
- Through atoms we build up formulas.
- We then use formulas to express various complex ideas.
- In this simple logic, an atom represents a declarative sentence that can be either true or false.
- An atom is created as a single unit.
- Its structure and composition are suppressed.
- However, there are many ideas that cannot be treated in this simple way.

First order logic

- Example
- The deduction of statements:
  - Every man is mortal.
  - Since Ade is a man, he is mortal.
- This reasoning is intuitively correct.

First order logic

- Example (cont.)
  - If we denote
    - P: Every man is mortal,
    - Q: Ade is a man,
    - R: Ade is mortal,
  - then R is not a logical consequence of P and Q within the framework of the propositional logic.
  - This is because the structures of P, Q, and R are not used in the propositional logic.

First order logic

- The first order logic has three more logical notions (called terms, predicates, and quantifiers) than does the propositional logic.
- Much of everyday and mathematical language can be symbolized by the first order logic.

First order logic

- Just as in the propositional logic, we first have to define atoms in the first-order logic.
- Example
  - We want to represent "*x* is greater than 3."
  - We first define a predicate *GREATER*(*x*,*y*) to mean "*x* is greater than *y*."
  - A predicate is a relation.
  - Then the sentence "*x* is greater than 3" is represented by *GREATER*(*x*,3).

First order logic

- Example:
  - Similarly, we can represent "x loves y" by the predicate *LOVE*(x,y).
  - Then "John loves Mary" can be represented by *LOVE*(John, Mary).

First order logic

- We can also use function symbols in the first order logic.
- Example
  - We can use *plus*(*x*,*y*) to denote "*x*+*y*" and *father*(*x*) to mean the father of *x*.
  - The sentences "*x*+1 is greater than *x*" and "John's father laves John" can be symbolized as
    - *GREATER*(*plus*(*x*,1),*x*)
    - *LOVE*(*father*(John),John).

First order logic

- Atoms:
  - *GREATER*(*x*,3)
  - *LOVE*(John,Mary)
  - *GREATER*(*plus*(*x*,1),*x*)
  - *LOVE*(*father*(John),John).
- Predicate symbols:
  - *GREATER*
  - *LOVE*.

First order logic

- In general, we are allowed to use the following four types of symbols to construct an atom:
  - i. Individual symbols or constants: These are usually names of objects, such as John, Mary, and 3.
  - ii. Variable symbols: These are customarily lowercase unsubscripted or subscripted letters, *x*, *y*, *z*, ...
  - iii. Function symbols: These are customarily lowercase letters f, g, h, ... or expressive strings of lowercase letters such as *father* and *plus*.
  - iv. Predicate symbols: These are customarily uppercase letters P, Q, *R*, ... or expressive strings of uppercase letters such as *GREATER* and *LOVE*.

First order logic

- Any function or predicate symbol takes a specified number of arguments.
- If a function symbol f takes n arguments, f is called an n-place function symbol.
- An individual symbol or a constant may be considered a function symbol that takes no argument.
- If a predicate symbol P takes n arguments, P is called an n-place predicate symbol.

First order logic

- Example
  - *father* is a one-place function symbol,
  - *GREATER* and *LOVE* are two-place predicate symbols.
- A function is a mapping that maps a list of constants to a constant.
- Example
- *father* is a function that maps a person named John to a person who is John’s father.

First order logic

- *father*(John) represents a person, even though his name is unknown.
- We call *father*(John) a term in the first order logic.

First order logic

- Definition. Terms are defined recursively as follows:
  - i. A constant is a term.
  - ii. A variable is a term.
  - iii. If f is an n-place function symbol, and $t_1, ..., t_n$ are terms, then $f(t_1, ..., t_n)$ is a term.
  - iv. All terms are generated by applying the above rules.

First order logic

- Example
  - Since x and 1 are both terms and *plus* is a two-place function symbol, *plus*(x,1) is a term according to the definition.
  - Furthermore, *plus*(*plus*(x,1),x) and *father*(*father*(John)) are also terms; the former denotes $(x+1)+x$, and the latter denotes the grandfather of John.

First order logic

- A predicate is a mapping that maps a list of constants to T or F.
- For example, *GREATER* is a predicate.
  - *GREATER*(*5,3*) is T,
  - but *GREATER*(1,3) is F.

First order logic

- Definition
  - If P is an n-place predicate symbol, and $t_1, \ldots, t_n$ are terms, then $P(t_1, \ldots, t_n)$ is an *atom*.
- Once atoms are defined, we can use the same five logical connectives as in the propositional logic to build up formulas.
- Furthermore, since we have introduced variables, we use two special symbols $\forall$ and $\exists$ to characterize variables.
- The symbols $\forall$ and $\exists$ are called, respectively, the *universal* and *existential quantifiers*.

First order logic

- If x is a variable, then $(\forall x)$ is read as "for all x", "for each x" or "for every x", while $(\exists x)$ is read as "there exists an x" "for some x" or "for at least one x"
- Example
  - Symbolize the following statements:
    - (a) Every rational number is a real number.
    - (b) There exists a number that is a prime.
    - (c) For every number x, there exists a number y such that $x<y$.

First order logic

- Denote “x is a prime number” by $P(x)$, “x is a rational number” by $Q(x)$, “x is a real number” by $R(x)$, and “x is less than y” by $LESS(x,y)$.
- Then statements can he denoted, respectively, as
  - (a’) $(\forall x)(Q(x) \Rightarrow R(x))$
  - (b’) $(\exists x)P(x)$
  - (c’) $(\forall x)(\exists y)LESS(x,y)$.
- Each of the expressions (a’), (b’), and (c’) is called a formula.

First order logic

- The *scope of a quantifier* occurring in a formula - the formula to which the quantifier applies.
- Example
  - The scope of both the universal the existential quantifiers in the formula $(\forall x)(\exists y)LESS(x,y)$ is $LESS(x,y)$.
  - The scope of the universal quantifier in the formula $(\forall x)(Q(x) \Rightarrow R(x))$ is $(Q(x) \Rightarrow R(x))$.

First order logic

- An occurrence of a variable in a formula is *bound* if and only if the occurrence is within the scope of a quantifier employing the variable, or is the occurrence in that quantifier.
- An occurrence of a variable in a formula is *free* if and only if this occurrence of the variable is not bound.

First order logic

- A variable is *free* in a formula if at least one occurrence of it is free in the formula.
- A variable is *bound* in a formula if at least one occurrence of it is bound.
- Example
  - In the formula $(\forall x)P(x,y)$, since both the occurrences of x are bound, the variable x is bound.
  - The variable y is free since the only occurrence of it is free.

First order logic

- A variable can be both free and bound in a formula.
- Example
  - *y* is both free and bound in the formula $(\forall x)P(x,y) \wedge (\forall y)Q(y)$.

First order logic

- Definition
  - Well formed formulas, or formulas for short, in the first-order logic are defined recursively as follows:
    - An atom is a formula. (Actually, “atom” is an abbreviation for an atomic formula.)
    - If F and G are formulas, then $\sim(F)$, $(F \vee G)$, $(F \wedge G)$, $(F \Rightarrow G)$ and $(F \Leftrightarrow G)$ are formulas.
    - If F is a formula and x is a free variable in F, then $(\forall x)F$ and $(\exists x)F$ are formulas.
    - Formulas are generated only by a finite number of applications of the three rules given above.

First order logic

- Parentheses may be omitted by the same conventions that hold in the propositional logic.
- The quantifiers have the least rank.
- Example
  - $(\exists x)A \vee B$ stands for $(((\exists x)A) \vee (B))$.

First order logic

- Example
  - Translate the statement “Every man is mortal. Ade is a man. Therefore, Ade is mortal.” into a formula.
    - Denote “x is a man” by $MAN(x)$, and “x is mortal” by $MORTAL(x)$. Then “every man is mortal” can be represented by
      - $(\forall x)(MAN(x) \Rightarrow MORTAL(x))$,
    - “Ade is a man” by
      - $MAN(\text{Ade})$, and
    - “Ade is mortal” by
      - $MORTAL(\text{Ade})$.

First order logic

- Example (cont.)
  - The, whole statement can now be represented by
    - $(\forall x)(MAN(x) \Rightarrow MORTAL(x)) \wedge MAN(\text{Ade}) \Rightarrow MORTAL(\text{Ade})$.

First order logic

- In the propositional logic, an interpretation is an assignment of truth values to atoms.
- In the first-order logic, since there are variables involved, we have to do more than that.
- To define an interpretation for a formula in the first-order logic, we have to specify:
  - the domain
  - an assignment to constants, function symbols, and predicate symbols occurring in the formula.

First order logic

- For every interpretation of a formula over a domain D, the formula can be evaluated to T or F according to the following rules:
  - If the truth values of formulas G and H are evaluated, then the truth values of the formulas $\sim G$, $(G \wedge H)$, $(G \vee H)$, $(G \Rightarrow H)$, and $(G \Leftrightarrow H)$ are evaluated by using the rules that hold in the propositional logic.
  - $(\forall x)G$ is evaluated to T if the truth value of G is evaluated to T for every x in D; otherwise, it is evaluated to F.
  - $(\exists x)G$ is evaluated to T if the truth value of G is T for at least one x in D; otherwise, it is evaluated to F.

First order logic

- Any formula containing free variables cannot be evaluated.
- It should be assumed either that formulas do not contain free variables, or that free variables are treated as constants.

First order logic

- Example
  - Let us consider the formulas
    - $(\forall x)P(x)$ and $(\exists x){\sim}P(x)$.
    - Let an interpretation be as follows:
      - Domain: $D=\{1,2\}$.
      - Assignment for P: $P(1)=T$, $P(2)=F$.
- $(\forall x)P(x)$ is F in this interpretation because $P(x)$ is not T for both $x=1$ and $x=2$.
- Since ${\sim}P(2)$ is T in this interpretation, $(\exists x){\sim}P(x)$ is T in this interpretation

First order logic

- Example
  - The formula
    - $(\forall x)(\exists y)P(x,y)$
  - The interpretation
    - $P(1,1)=T$, $P(1, 2)=F$, $P(2,1)=F$, $P(2,2)=T$,
  - If $x=1$, there is a y, 1, such that $P(1,y)$ is T.
  - If $x=2$, there is also a y, 2, such that P(2,y) is T.
- Therefore, in this interpretation, for every x in D, there is a y such that $P(x,y)$ is T.
- That means that $(\forall x)(\exists y)P(x,y)$ is T in this interpretation.

First order logic

- The formula
  - G: $(\forall x)(P(x) \Rightarrow Q(f(x), a))$.
- There are one constant a, one one-place function symbol f, one one-place predicate symbol P, and one two-place predicate symbol Q in G.

First order logic

- The interpretation I of G
  - Domain: $D=\{1,2\}$.
  - Assignment for a: $a=1$.
  - Assignment for f: $f(1)=2$, $f(2)=1$.
  - Assignment for P and Q:
    - $P(1)=F$, $P(2)=T$, $Q(1,1)=T$, $Q(1,2)=T$, $Q(2,l)=F$, $Q(2,2)=T$.

First order logic

- If x=1, then
  - $P(x) \Rightarrow Q(f(x),a) = P(1) \Rightarrow Q(f(1),a)$
  - $= P(1) \Rightarrow Q(2,1)$
  - $= F \Rightarrow F = T$.
- If x=2, then
  - $P(x) \Rightarrow Q(f(x),a) = P(2) \Rightarrow Q(f(2),a)$
  - $= P(2) \Rightarrow Q(1,1)$
  - $= T \Rightarrow T = T$.

First order logic

- Since $P(x) \Rightarrow Q(f(x),a)$ is true for all elements x in the domain D, the formula $(\forall x)(P(x) \Rightarrow Q(f(x),a))$ is true under the interpretation I.

First order logic

- Exercise
  - Evaluate the truth values of the following formulas under the interpretation given in the previous example.
    - (a) $(\exists x)(P(f(x)) \wedge Q(x,f(a))$
    - (b) $(\exists x)(P(x) \wedge Q(x,a))$
    - (c) $(\forall x)(\exists y)(P(x) \wedge Q(x,y))$

First order logic

- Once interpretations are defined, all the concepts, such as validity, inconsistency, and logical consequence, defined in propositional logic can be defined analogously for formulas of the first-order logic.

First order logic

- Definition
  - A formula G is *consistent* (*satisfiable*) if and only if there exists an interpretation I such that G is evaluated to T in I. If a formula G is T in an interpretation I, we say that I is a *model* of G and I *satisfies* G.

First order logic

- Definition
  - A formula G is *inconsistent* (*unsatisfiable*) if and only if there exists no interpretation that satisfies G.
- Definition
  - A formula G is *valid* if and only if every interpretation of G satisfies G.
- Definition
  - A formula G is a *logical consequence* of formulas $F_1, F_2,\ldots, F_n$, if and only if for every interpretation I, if $F_1 \wedge F_2 \ldots \wedge F_n$ is true in I, G is also true in I.

First order logic

- The relations between validity (incon-sistency) and logical consequence that hold in propositional logic also hold for the first-order logic.
- In fact, the first-order logic can be considered as an extension of the propositional logic.
- When a formula in the first-order logic contains no variables and quantifiers, it can be treated just as a formula in the propositional logic.

First order logic

- In the first-order logic, since there are an infinite number of domains, in general, there are an infinite number of inter-pretations of a formula.
- Therefore, unlike in the propositional logic, it is not possible to verify a valid or an inconsistent formula by evaluating the formula under all the possible inter-pretations.

First order logic

- Exercise
  - The formulas
    - F_1: $(\forall x)(P(x) \Rightarrow Q(x))$
    - F_2: $P(a)$
  - Prove that formula $Q(a)$ is a logical consequence of F_1 and F_2.