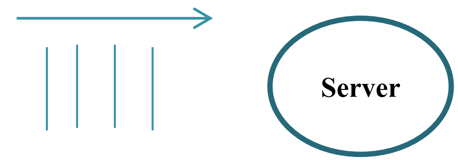
M/M/1 model



- Service rate = μ/h
- Arrival rate = λ/h
- System does not take into account the earlier states of the system in order to define the present state of this system. (memoryless property)
- only one event will take place in a small interval of time, h
  - Either arrival or service
- At steady state, we want to know the probability, that the system has certain customers at a point in time
- customer is interested in:
  - L_s: length of the system (expected number of people who are actually in the system, including the person who is being served)
  - L_q : length of the queue (expected number of people who are waiting for service in this system)
  - W_s: waiting time in the system
  - W_q : waiting time in the queue.
- If we derive an expression for one, we can derive an expression for all
- We are interestee in expression for the steady state probabilities that there are 0, 1, 2, 3…. People in the system. i.e. $P_{0,}$ P_2, P_3, etc

M/M/1 model (contd)

- Probability that there are n people in the system at time t+h is given as:
- $P_n(t+h) = P_{n-1}(t)$ * Probability of 1 arrival and no service $+P_{n+1}(t)$ * probability of no arrival and one service + $P_n(t)$ * probability of no arrival and no service)
- Note that we are not considering other probabilities due to the assumption that in a small interval, only one event can take place e.g we will not consider probabiity of one arrivan and one service or probability of 2 arrivals)
  - P(one arrival = λh) $\rightarrow$ P(no arrival) = 1 – λh
  - P(one service = μh) $\rightarrow$ P(no service) = 1 - μh

M/M/1 model (contd)

$$P_n(t+h) = P_{n-1}(t) * \lambda h(1-\mu h) + P_{n+1}(t)\mu h(1-\lambda h)$$
$$+ P_n(t)(1-\lambda h)(1-\mu h)$$

$$P_n(t+h) = P_{n-1}(t)*(\lambda h - \lambda\mu h^2) + P_{n+1}(t)(\mu h-\lambda\mu h^2) + P_n(t)(1- \mu h-\lambda h+ \lambda\mu h^2)$$

We leave out the higher order atoms i.e $\lambda\mu h^2 =0$

$$P_n(t+h) = P_{n-1}(t)\lambda h + P_{n+1}(t)\mu h + P_n(t)(1- \lambda h - \mu h)$$
$$= P_{n-1}(t)\lambda h + P_{n+1}(t)\mu h + P_n(t) - P_n(t)\lambda h - P_n(t)\mu h$$

$$P_n(t+h) – P_n(t) = P_{n-1}(t)\lambda h + P_{n+1}(t)\mu h – h P_n(t)(\lambda+ \mu)$$

Dividing through by h

$$\frac{P_n(t+h) – P_n(t)}{h} = P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)(\lambda+\mu)$$

M/M/1 model (contd)

Now, for steady state, probability is not going to be time dependent therefore

$$\frac{P_n(t+h) - P_n(t)}{h} = 0$$

Hence for steady state:

$$0 = P_{n-1}\,\lambda + P_{n+1}\,\mu - P_n\,(\lambda+\mu)$$

$$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu)P_n \qquad \text{……………………. (1)}$$

M/M/1 model (contd)

- Now,
- Probability of 0 person in the system at time t+h is given as:
- $P_0(t+h) = P_1(t)$ * (probability of no arrival and one service) + $P_0(t)$ * (probability of no arrival and no service)

$$P_0(t+h) = P_1(t)(1-\lambda h)\mu h + P_0(t)(1-\lambda h)1$$

(note that probability of no service is 1)

- Expanding and leaving out the higher order

$$P_0(t+h) = P_1(t)\mu h + P_0(t) - P_0(t)\lambda h$$

$$\frac{P_0(t+h) - P_0(t)}{h} = P_1(t)\mu - P_0(t)\lambda$$

At steady state, h = 0 therefore

$$\mu P_1 = \lambda P_0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Recall:

$\lambda P_{n-1} + \mu P_{n+1} = (\lambda + \mu)P_n$ ………………….. (1)

$\mu P_1 = \lambda P_0$ ………………………… (2)

From equation (2), we can derive

$$P_1 = \frac{\lambda}{\mu} P_0$$

Substituting in equation (1)

$\lambda P_0 + \mu P_2 = (\lambda + \mu)P_1$

$\lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1$

But $\mu P_1 = \lambda P_0$

$\lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1$

But from (2), $\mu P_1 = \lambda P_0$

Hence

$\lambda P_0 + \mu P_2 = \lambda P_1 + \lambda P_0$

- $\mu P_2 = \lambda P_1$
- $P_2 = (\frac{\lambda}{\mu}) P_1$
- And we know that $P_1 = \frac{\lambda}{\mu} P_0$
- Therefore $P_2 = (\frac{\lambda}{\mu})^2 P_0$

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- Let $p = \frac{\lambda}{\mu}$

So we have:

$P_1 = pP_0$

$P_2 = pP_1 = p^2P_0$

In a similar manner we derive:

$P_3 = pP_2 = p^3P_0$

Hence:

$P_n = p^nP_0$

We still don't know the actual values of

$P_1, P_2, \ldots\ldots P_n$

Note that they are all dependent on P_0

Hence we need to know the value of P_0

Recall that sum of steady-state probabilities = 1

Hence $P_0 + P_1 + \ldots.. + P_\infty = 1$

$P_0 + pP_0 + p^2P_0 + p^3P_0 + \ldots + \infty = 1$

$P_0(1 + p + p^2 + p^3 + \ldots + p^\infty) = 1$

Note that this is an infinite series and a geometric series

We know that the sum to infinite of a geometric series = 1-r provided r<1

And $p = \dfrac{\lambda}{\mu} < 1$ for infinite population model

When $p = \frac{\lambda}{\mu} < 1$, we can apply the infinite geometric series summation formula

$1 - p = 1$ from which

$$P_o\left(\frac{1}{1-p}\right) = 1$$

$P_o = 1-p$

Now $p_1 = pP_0 = p(1-p)$

In general terms,

$P_n = p^n P_0 = p^n (1-p)$

We know that $p = \frac{\lambda}{\mu}$

Therefore we can calculate $P_0, P_1, P_2, \ldots P_n$.

Note that that the inputs are λ, μ and c

And for M/M/1 model, c=1

Hence If we know *p*, then we can find out the expression for P_0 , P_1 … P_n

we are also interested in the expressions for W_s, W_q, L_s and L_q in terms of P_0, P_1, etc, The expected number of people in the system L_s is given as:

$L_s = \sum_{j=0}^{\infty} jP_j = \sum jp^j P_0$ Where j = the number of people in the system

$$= P_0 p \sum jp^{j-1}$$

$$= P_0 p \sum \frac{d}{d_p} p^j$$

$$P_0 p \frac{d}{d_p}(1 + p + p^2 + \cdots \infty)$$

$$= P_0 p \frac{d}{d_p}(\frac{1}{1-p})$$

$$= P_0 p \frac{d}{d_p} \frac{1}{(1-p)^2} = \frac{(1-p)p}{(1-p)^2} = \frac{p}{(1-p)}$$

Hence Ls = expected number of people in the system $= \frac{p}{(1-p)}$

$$L_s = \frac{p}{1-p}$$

But $L_s = L_q + \frac{\lambda}{\mu}$

$Ls = \lambda W_s$

$L_q = \lambda W_q$

--- Little's Law