Automata theory and formal languages

Properties of regular languages

Operations that preserve regularity

- In the last lecture we saw three operations that preserve regularity:
  - Union: If L, L' are regular languages, so is $L \cup L$'
  - Concatenation: If L, L' are regular languages, so is LL'
  - Star: If L is a regular language, so is $L*$

- Exercise: If L is regular, is L^4 also regular?
- Answer: Yes, because

$$L^4 = ((LL)L)L$$

Example

- The language L of strings that end in 101 is regular

$$(0+1)^*101$$

- How about the language $\overline{L}$ of strings that do not end in 101?

Example

- **Hint:** A string does not end in 101 if and only if it ends in one of the following patterns:

$$000,\ 001,\ 010,\ 011,\ 100,\ 110,\ 111$$

  (or it has length 0, 1, or 2)

- So $\overline{L}$ can be described by the regular expression

$$(0+1)^*(000+001+010+010+100+110+111)$$
$$+\ \varepsilon + (0 + 1) + (0 + 1)(0 + 1)$$

Complement

- The complement $\overline{L}$ of a language L is the set of all strings (over Σ) that are not in L

- Examples ($\Sigma = \{0, 1\}$)
  - L_1 = all strings that end in 101
  - $\overline{L_1}$ = all strings that do not end in 101
    = all strings end in $000, \ldots, 111$ or have length 0, 1, or 2

  - $L_2 = 1^* = \{\varepsilon, 1, 11, 111, \ldots\}$
  - $\overline{L_2}$ = all strings that contain at least one 0
    $= (0 + 1)^*0(0 + 1)^*$

Closure under complement

- If L is a regular language, is $\overline{L}$ also regular?
- Previous examples indicate answer should be yes

- Theorem

> If L is a regular language, so is $\overline{L}$.

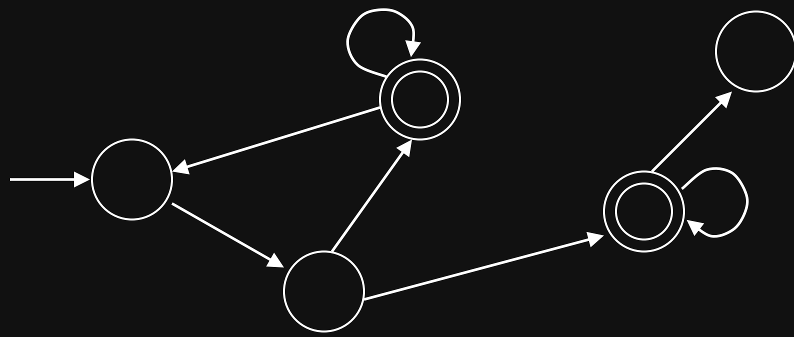
Proof of closure under complement

- To prove this in general, we can use any of the equivalent definitions for regular languages:

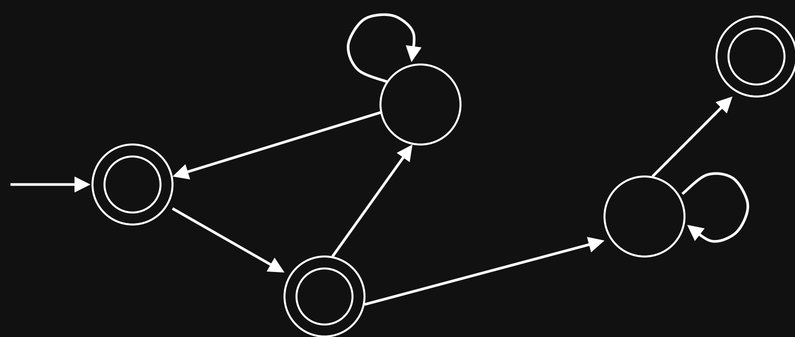
- In this proof DFA definition will be most convenient
  - We will assume L is accepted by a DFA, and show the same for $\overline{L}$

Proof of closure under complement

- Suppose L is regular, then it is accepted by a DFA M



- Now consider the DFA M' with the accepting and rejecting states of M reversed



Proof of closure under complement

- Now for every input $x \in \Sigma^*$:

M accepts x

$\Updownarrow$

After processing x, M ends in an accepting state

$\Updownarrow$

After processing x, M' ends in an rejecting state

$\Updownarrow$

M' rejects x

} Language of M' is $\overline{L}$

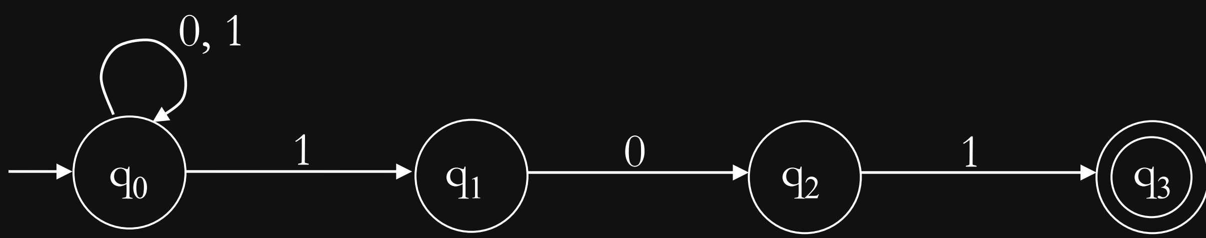
$\Downarrow$

$\overline{L}$ is regular

$\square$

A warning

- NFA for language of strings ending in 101



- Give NFA that accepts strings that do not end in 101

Intersection

- The intersection $L \cap L'$ is the set of strings that are in both L and L'

- Examples:

  $L = (0 + 1)^*111$ $\quad$ $L' = 1^*$ $\quad$ $L \cap L' = ?$

  $L = (0 + 1)^*101$ $\quad$ $L' = 1^*$ $\quad$ $L \cap L' = ?$

- If L, L' are regular, is $L \cap L'$ also regular?

Closure under intersection

- Theorem

> If L and L' are regular languages, so is $L \cap L'$.

- Proof: L regular $\Downarrow$ $\overline{L}$ regular; L' regular $\Downarrow$ $\overline{L'}$ regular

  $\overline{L} \cup \overline{L'}$ regular

  But $\overline{L} \cup \overline{L'} = \overline{L \cap L'}$

  $\overline{L \cap L'}$ regular $\Downarrow$ $L \cap L'$ regular.

$\square$

Reversal

- The reversal w^R of a string w is w written backwards

$$w = \text{cave} \qquad\qquad w^R = \text{evac}$$

- The reversal L^R of a language L is the language obtained by reversing all its strings

$$L = \{\text{push}, \text{pop}\} \qquad\qquad L^R = \{\text{hsup}, \text{pop}\}$$

Reversal of regular languages

- $L =$ all strings that end in 101 is regular

$$(0+1)^*101$$

- How about L^R?
- This is the language of all strings beginning in 101
- Yes, because it is represented by

$$101(0+1)^*$$

Closure under reversal

- Theorem

  If L is a regular language, so is L^R.

- Proof

  – We will use the representation of regular languages by regular expressions

Proof of closure under reversal

- If L is regular, then there is a regular expression E that describes it
- We will give a systematic way of reversing E
- Recall that a regular expression can be of the following types:
  - Special expressions $\emptyset$ and ε
  - Alphabet symbols a, b, $\ldots$
  - The union, concatenation, or star of simpler expressions
- In each of these cases we show how to do a reversal

Proof of closure under reversal

regular expression E	reversal E^R
$\varnothing$	$\varnothing$
ε	ε
a (alphabet symbol)	a
$E_1 + E_2$	$E_1{}^R + E_2{}^R$
$E_1 E_2$	$E_2{}^R E_1{}^R$
$E_1{}^*$	$(E_1{}^R)^*$